Ruin Probabilities with Dependent Claims

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ABSTRACT

Determining the probability of ruin and time to ruin are important problems in classical risk theory. Historically, it has been assumed that the individual claims are independent from each other. However, it is known that in reality this is not the case. Claims generated by one policy in an insurance portfolio may induce claims from other policies within the same portfolio.

In this contribution, we investigate the effects of dependence on both the probability of ruin and time to ruin. A dependence structure on claim occurrence is introduced through copulas. A simulation study is then performed to examine the effects of different dependence structures on the total number of claims, the probability of ruin and the time to ruin. It is found that whilst the probability of ruin increases as dependence upon claim occurrence, as measured by Kendall's tau, increases, there is little effect on the time to ruin.

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All this comes from God. *Soli Deo gloria*.

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Chapter 1

Introduction

The purpose of our investigation is to examine the surplus process, with a particular emphasis on the concept of ruin. We begin with an overview of the model, which provides the motivation for our investigation.

Amongst other things, insurance companies exist to pool together risks faced by individuals such that those who may experience a loss at any one time receives compensation that would help alleviate the financial consequences of the loss. Because of this, it is vital for insurance companies to set aside an amount of money, as *reserves* or *surplus*, so that it would be able to meet its commitments to pay claims whenever they occur.

Over time, an insurance company may be able to accumulate a surplus. This happens when the premiums collected over a time period exceed the claims that have been paid over that period.[1] This surplus could then be used by the insurance company to shield against future time periods when the claims paid exceed the premiums collected.

Initially, however, an insurance company needs to set aside additional reserves for a line of business. This is because there is always insufficient funds generated from earlier periods of the operation to provide for a sufficient buffer

[1]While in accounting the concept of surplus is more complex, for our purposes this definition will suffice.

for future claims. In order to prevent *ruin*, where the claims paid exceed the reserves available, the company must provide sufficient initial capital at the beginning and carefully monitor its reserves level throughout its operations. In addition, the company should also arrange the appropriate type of reinsurance to mitigate the effect of large claims.

This situation is further complicated by the nature of the losses that are covered by general insurance. Although there would be a large number of small claims, there would also be some extremely large claims that are several orders of magnitude greater than the mean claim amount. Thus it is even more important that the surplus is modelled, predicted and managed properly to ensure the continual survival and profitability of the company.

The classical risk model is a particularly useful tool in modelling the surplus process. Its utility comes from its flexibility, in which each component in the model—premiums, claim numbers and claim amounts—can be modelled separately. The entire line of business can be modelled as a whole instead of on a policy-by-policy basis. Hence the classical risk model has widespread applications in general insurance decision-making.

1.1 The Classical Risk Model

The classical risk model is widely studied in the actuarial literature. Philipson (1968) provided an extensive survey of the fundamental developments of the theory, along with a large number of references.

Before embarking on a formal definition of the elements of the classical risk model, consider an insurance system from an intuitive point of view.

Consider the major cash flows that would affect the operations of an insurance company. These would include premiums, claims, expenses, and reinsurance cash flows. To simplify this cash flow model, we make the following adjustments. Firstly we subtract the expenses paid from the premiums received, so that the premiums collected are net of expenses. Next we take reinsurance inflows and outflows into account when we consider premiums and claims, so that the premiums received are net of reinsurance premiums, and the claims paid are only those that are paid by the direct insurer. These adjustments leave us with two cash flows: premiums and claims.

1.1.1 The Premium Process

Consider $\{\Pi(t)\}$, the premiums collected by the company during the time period $(0, t]$, less any expenses and reinsurance outflows. Note that this inflow of premiums is different from "earned premiums" in the accounting sense.

While the premium rate is usually dependent on the expected claim amount plus some premium loading factor θ, for convenience we shall assume that premiums are collected continuously by the company at a constant rate of c per time period, thus $\Pi(t) = ct$. This assumption has been widely used in the literature (Bühlmann 1970, 136).

1.1.2 The Aggregate Claims Process

In the individual risk model, it is assumed that the number of policyholders at time s, $n(s)$ is known. Then for each time period t the total claim amount for $(0, t]$ can be expressed as

$$S(t) = \sum_{i=1}^{n(t)} Y_i(t), \tag{1.1.1}$$

where the $Y_i(t)$ represent the claims made by policyholder i over time $(0, t]$, or zero if the policyholder did not claim at time $(0, t]$. Further, the $Y_i(t)$ can further be decomposed into

$$Y_i(t) = I_i(t)X_i(t), \tag{1.1.2}$$

where $I_i(t) = 1$ if the policyholder claims over time $(0, t]$, and zero otherwise, and $X_i(t)$ is the claim amount if the policyholder claimed. Traditionally, it is *assumed* that the $X_i(t)$ are independent and identically distributed random variables with distribution $F(x)$.

The $X_i(s)$ are commonly modelled as the exponential, gamma, lognormal, Pareto or Weibull random variables. Of these distributions, only the exponential is light-tailed, in the sense that $1 - F(x)$ decreases faster than e^{zx} for all $z > 0$. All the other distributions are heavy-tailed. The exponential distribution is used for $F(x)$ because it produces tractable results when computing ruin probabilities, as shown in the next chapter. However, heavy-tailed distributions are more realistic in modelling claim amounts, especially those from general insurance. Results when the claim sizes are heavy-tailed can be found in, for example, Rolski, Schmidli, Schmidt, and Teugels (1999).

Because in practice there may not be sufficient data to model claims using the individual risk model, the *collective risk model* is used. In the collective risk model, the claim payments are modelled using a compound stochastic process, in that both the timing of the claim (and hence the number of claims in a time period) and the amount of each claim are both stochastic.

The *claim number process*, $\{N(t)\}$, represents the total number of claims that have occurred during the time period $(0, t]$, and clearly must take on nonnegative integer values. This replaces the deterministic $n(t)$ in (1.1.1) above.

While any counting process can be used for $\{N(t)\}$, the Poisson process, with some rate λ, is the most common. The utility of using the Poisson process is its simplicity, and that any counting process with independent increments and monotonically increasing mean value function can be converted to a Poisson process, through the introduction of a new time variable (see example (d) in Feller 1966, 178).

The *claim amount process*, $\{X_i > 0 : i = 1, 2, \ldots, N(t)\}$, represents the claim amount of each of the $N(t)$ claims that have incurred in $(0, t]$. This replaces the $X_i(s)$ in (1.1.1) above. The assumptions regarding independence of the X_i are the same as in the individual risk model. Thus under the collective risk model, $S(t)$ becomes

$$S(t) = \sum_{i=0}^{N(t)} X_i. \tag{1.1.3}$$

In the case where $\{N(t)\}$ is a Poisson process, $\{S(t)\}$ is known as a *compound Poisson process*. This process is commonly used in the literature as it leads to tractable results, e.g. in the formulation of ruin probabilities. Further, when two compound Poisson processes are added, it result itself is a compound Poisson process (Daykin et al. 1994, 66).

Gerber (1984) and Michel (1987) have shown that the collective risk model is a good approximation to the individual risk model. Each has derived an upper bound for the total variation distance between the distribution of $S(t)$ under each of the two models.

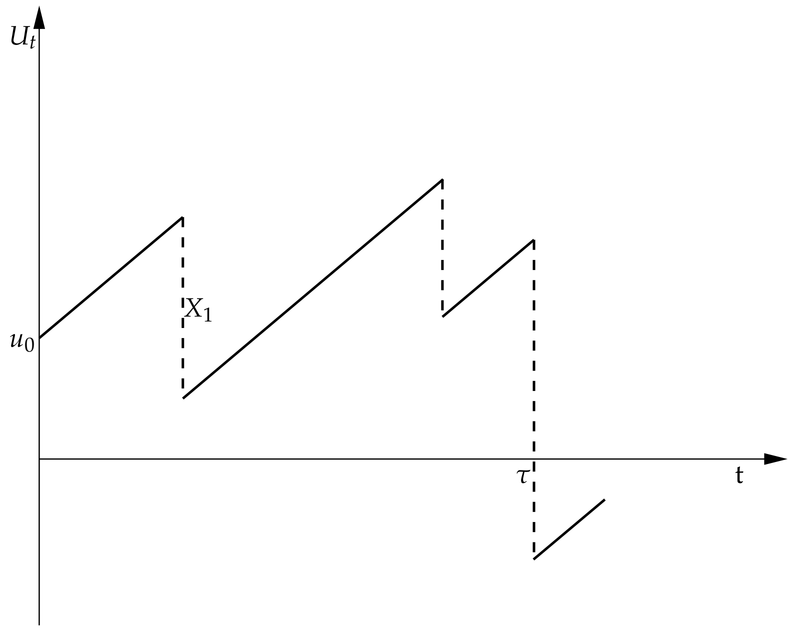

Figure 1.1: A realisation of a surplus process.

1.1.3 The Surplus Process

Suppose there exists an initial reserve of u_0 at $t = 0$. Then the reserve (or surplus) at time t, $U(t)$, would be

$$U(t) = u_0 + \Pi(t) - S(t). \tag{1.1.4}$$

That is, the reserve up to time t is the initial reserve, plus all the premiums received up to time t, minus all the claims that have been paid up to time t.

$\{U(t)\}$ is then the *surplus process*, a typical realisation of which is illustrated in Figure 1.1 for $\Pi(t) = ct$. The diagram labels the initial capital u_0, and the amount of the first claim X_1.

1.2 Ruin

As noted before, ruin occurs when the claims paid exceed the surplus available. Mathematically, this is when $U(t) < 0$. Thus we define the time to ruin τ as $\tau = \inf_t \{t : U(t) < 0\}$. If $U(t) \geq 0$ for all $t > 0$, then we define $\tau = \infty$. The time to ruin is illustrated in Figure 1.1.

This enables us to easily define the probabilities of ruin. The infinite-horizon (or ultimate) ruin probability would be

$$\psi(u) = P(\tau < \infty \mid u_0 = u),$$

and the finite-horizon ruin probability would be

$$\psi(u, t) = P(\tau < t \mid u_0 = u).$$

These definitions follow the ones used in Rolski et al. (1999). A slightly less vigorous, but equivalent, set of definitions can be found in Beard, Pentikäinen, and Pesonen (1969) and Dickson and Waters (1992).

1.3 Limitations of the Classical Model

The classical model is not without its limitations. Taylor and Buchanan (1988) commented that

> While this theory is well developed and well known, there are a number of respects in which it lacks realism to a point which militates against its practical use without substantial modification.

Nevertheless, Philipson (1968) attributed the substantial body of results that came from this theory to the fact that these assumptions are made by the early researchers.

The main source of the limitations stem from the assumptions that are placed on the claim frequency and claim size distributions. Of particular interest to us is the assumption that the increments in this surplus process are independent.

Dependence within the risk model can occur at several levels.

1. the increments within the claim frequency process $\{N(t)\}$, or equivalently the $I_i(t)$ may be dependent

2. the claim amounts X_i themselves may be dependent.

It is not difficult to envisage situations where these would hold.

1. In home and contents insurance where storm-water damage is covered, the rate of occurrence would be affected by climate patterns. It is well-known that climate patterns in eastern Australia is heavily dependent on the Southern Oscillation Index, which fluctuates every few years (for example, Kiladis and Diaz 1989). Thus it is clear that $\{N(t)\}$ does not have independent increments.

2. Again, the claim amounts themselves may also be dependent. Continuing with our example of home and contents insurance, a storm in a single area would cause similar damage to the properties in that area, consequently generating claims of similar amounts.

Clearly it can be seen that the assumption of independent increments would be inadequate in modelling real-world insurance processes.

Dependence within the model would have effects on the distribution of the aggregate claims $\{S(t)\}$, and consequently, the probability of ruin $\psi(u)$ and the time to ruin τ. Hence it would be important for dependence to be considered in general insurance decision-making: when determining the initial reserves required for a new line of business, examining the adequacy of the current reserves, and acquiring appropriate forms of reinsurance.

1.4 The Purpose of the Thesis

The purpose of the thesis, therefore, is to investigate the effects of dependence on ruin: both the probability of ruin and the time to ruin. Given that it is extremely difficult to obtain analytical results for these quantities, especially after the introduction of dependence, these effects are examined through a series of Monte Carlo simulations. The dependence within the models is introduced through structures called *copulas*, which is introduced in chapter 3.

The thesis is organised as follows. Chapter 2 reviews some of the key results developed in classical risk theory. In addition, it surveys some recent developments in the literature regarding dependence within the risk model. An investigation of the effects of dependence on ruin, using Monte Carlo techniques, are performed in chapter 3. We conclude in chapter 4.

CHAPTER 2

REVIEW OF KEY RESULTS IN RISK THEORY

In this chapter we review some key results from risk theory, survey some of the recent developments in the literature into the role dependence plays in the model, and evaluate its impact.

2.1 Classical Results in the Computation of Ruin Probabilities

Assume that $N(t)$ in (1.1.3) is a Poisson process, $\Pi(t) = ct$, and the X_i are i.i.d. random variables. Thus the aggregate claims process within the surplus process would be a compound Poisson process earlier defined.

2.1.1 The Integro-differential Equation

Given the above assumptions, the probability of ruin $\psi(u)$ satisfies the well-known integro-differential equation:

$$\psi'(u) = \frac{\lambda}{c}\psi(u) - \frac{\lambda}{c}\int_0^u \psi(u-x)\,dF(x) - \frac{\lambda}{c}(1 - F(u)). \tag{2.1.1}$$

Here λ is the parameter in the Poisson process, c is the premium rate in (1.1.4) and $F(x)$ is the common distribution function of the individual claim amounts, with $dF(x) = F'(x)dx$ if $F(x)$ is absolutely continuous.

The proof of this can be found in, for example, Dickson and Waters (1992) and Klugman et al. (1998). The proof involves conditioning on the event of a claim occurring in a small interval h, which produces a continuous form of a difference equation for $\psi(u)$, then taking limits on $h \to 0$.

A solution to (2.1.1) can be found using Laplace transforms. It can be shown (e.g. Rolski et al. 1999, 165) that if $L_\psi(s) = \int_0^\infty e^{-s\psi(u)}\, du$ is the Laplace transform of $\psi(u)$,

$$L_\psi(s) = \frac{1}{s} - \frac{c - \lambda\mu}{cs - \lambda(1 - L_F(s))}, \tag{2.1.2}$$

where $L_F(s)$ is the Laplace-Stieltjes transform of F (if F is absolutely continuous, this is equivalent to the Laplace transform of $F'(x)$) and $\mu = E(X_i)$.

The difficulty here rises when $L_\psi(s)$ is not of a type that the inverse can be found, which is generally the case. Consequently, it is extremely difficult to find a closed-form solution for $\psi(u)$, with some exceptions described later in this section.

2.1.2 The Adjustment Coefficient

The adjustment coefficient is particularly useful in the determination of both ruin probabilities and bounds of ruin probabilities. It is defined as the positive number r such that

$$\lim_{u\to\infty} \psi(u)e^{(r-\epsilon)u} = 0 \qquad \text{and} \qquad \lim_{u\to\infty} \psi(u)e^{(r+\epsilon)u} = \infty$$

hold for all $\epsilon > 0$.

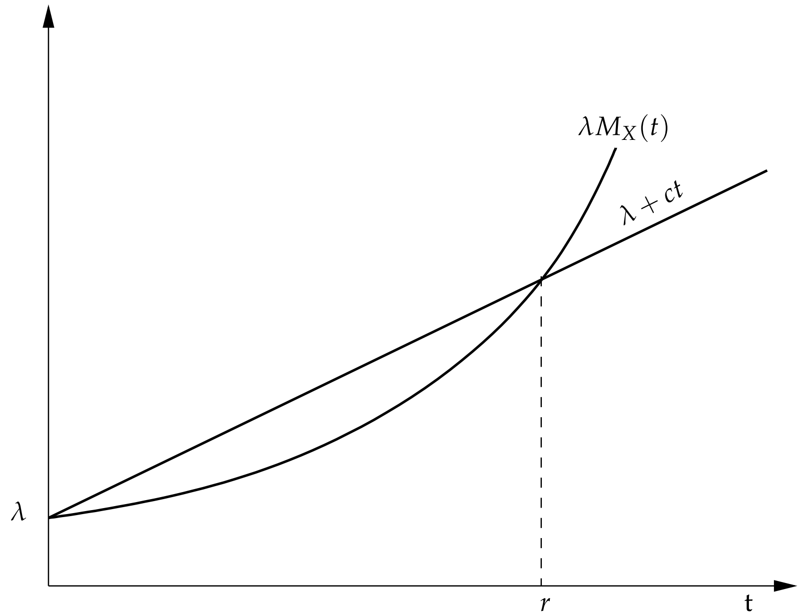

Figure 2.1: The adjustment coefficient.

For compound Poisson claim number processes, this is equivalent to the positive root of

$$\lambda M_X(r) = \lambda + cr, \tag{2.1.3}$$

where $M_X(r) = \int_0^\infty e^{rx}\, dF(x)$ is the moment generating function of F, evaluated at r. This root is illustrated in Figure 2.1.

It can be shown (Rolski et al. 1999, 170) that this coefficient exists when there exists $s' \in (-\infty, \infty]$ such that $M_X(s) < \infty$ whenever $s < s'$ and $\lim_{s \to s'-} M_X(s) = \infty$.

The adjustment coefficient is a rudimentary measure of risk in the collective, and will be used to determine the bounds of the probability of ruin.

One can use the adjustment coefficient to calculate the probability of ruin. If the adjustment coefficient r exists, then

$$\psi(u) = \frac{e^{-ru}}{E(e^{-rU_\tau}|\tau < \infty)}. \tag{2.1.4}$$

The proof of the above can be found in Bowers et al. (1997).

Because of the extreme difficulty in finding the denominator in (2.1.4), the Laplace transform is often the preferred way of calculating (analytically or numerically) the probability of ruin.

2.1.3 Closed-form Solutions

A closed form solution for $\psi(u)$ can be found if the X_i in (1.1.3) are i.i.d. exponential random variables, with parameter α. In this case,

$$\psi(u) = \frac{\lambda}{\alpha c} e^{-(\alpha-\lambda/c)u}. \tag{2.1.5}$$

This can be proved by firstly noting that $\mu = 1/\alpha$ and $L_F(s) = \alpha/(\alpha+s)$. Substituting into (2.1.2) gives

$$\begin{aligned} L_\psi(s) &= \frac{1}{s} - \frac{c-\lambda/\alpha}{cs - \lambda(1-\frac{\alpha}{\alpha+s})} \\ &= \frac{1}{s} - \frac{c-\lambda/\alpha}{cs - \frac{\lambda s}{\alpha+s}} \\ &= \frac{[c(\alpha+s)-\lambda] - (c-\lambda/\alpha)(\alpha+s)}{s[c(\alpha+s)-\lambda]} \\ &= \frac{\lambda}{\alpha}\frac{1}{c(\alpha+s)-\lambda} \\ &= \frac{\lambda}{\alpha c}\frac{1}{\alpha-(\lambda/c)+s}, \end{aligned}$$

whose inverse is (2.1.5), completing the proof.

Gerber et al. (1987) used a different transform to obtain an analytical solution to $\psi(u)$ for combinations of exponential distributions (also known as hyperexponential distributions). That is,

$$F(x) = 1 - \sum_{i=1}^{n} A_i e^{-\alpha_i x}$$

where $\sum A_i = 1$. This result is particularly useful as Feldmann and Whitt (1998) have shown that heavy-tail distributions whose density functions are completely monotone, i.e. $(-1)^n f^{(n)}(x) \geq 0$ for all $x > 0$ and $n = 1, 2, \ldots$, can be approximated arbitrarily closely by a hyperexponential distribution. Completely monotonic distributions include the Pareto, Weibull and exponential mixtures of inverse Gaussian (Abate and Whitt 1999), which are frequently used in practice to model claim amounts.

Dufresne (2001) has recently shown that if $L_F(s)$ in (2.1.2) is a rational function and $F(0) = 0$ (such as the Erlang distribution), then $\psi(u)$ can be expressed as

$$\psi(u) = 1 - \sum_{k=1}^{m} f_k(u) e^{-r_k u} \tag{2.1.6}$$

where the $f_k(u)$ are polynomials and the r_k are constants.

2.2 Lundberg Bounds on Ruin Probabilities

Because of the difficulty in obtaining a closed-form solution of the probability of ruin, we often resort to constructing bounds (especially the upper bound) of this $\psi(u)$. The most famous of these bounds is attributed to Lundberg (1930), who showed that if the adjustment coefficient exists, then

$$\psi(u) \leq e^{-ru}. \tag{2.2.1}$$

Asmussen and Nielsen (1995) proved a similar result when the premium rate is a right-continuous function of the reserve, i.e. $\Pi(t) = f(U(t)) * t$, by replacing the adjustment coefficient r with an adjustment coefficient function $r(u)$.

Taylor (1976) has used differential and integral inequalities to derive an improvement to (2.2.1). He showed that

$$\psi(u) \leq \left(\sup_{x\in[0,\omega)} \frac{e^{rx} \int_x^\infty \bar{F}(s)\, ds}{\int_x^\infty e^{rs} \bar{F}(s)\, ds} \right) e^{-ru}, \tag{2.2.2}$$

where $\bar{F}(x) = 1 - F(x)$ and $\omega = \sup\{x : F(x) < 1\}$, the maximum claim size. Further, he showed that the supremum is less than or equal to one, and thus this bound is an improvement to the classical Lundberg bound.

2.3 Approximations and Numerical Techniques

There are several well-known methods that has appeared in the literature to approximate the ruin probability. These include the Cramér-Lundberg approximation, moment filtering and numerical inversion.

The Cramér-Lundberg approximation uses the property that $\psi(u)$ approaches a limit as $u \to \infty$ (Rolski et al. 1999, 172f.), to provide an approximation for $\psi(u)$ of

$$\tilde{\psi}(u) = \frac{c - \lambda u}{\lambda M_X(r) - c} e^{-ru}. \tag{2.3.1}$$

It is worthy to note that when the X_i are exponentially distributed (2.1.5) is equivalent to (2.3.1).

Another method to approximate the ruin probability is to use moment filtering. The general idea is to substitute one process with another process whose ruin probability can be easily found. De Vylder (1978) substituted $U(t)$ with $\tilde{U}(t)$ which has exponentially distributed claim amounts, although he has experienced less than ideal results when approximating ruin probabilities with log-normal distributed claim sizes. Beekman (1969) substituted $Z(t) = 1 - \psi(t)$ with $\tilde{Z}(t)$, having a gamma distribution, which then provides an approximation of $\tilde{\psi}(u) = 1 - \tilde{Z}(t)$.

Finally, one can numerically invert the Laplace transform given in (2.1.2). Using the formula in section 3 of Abate and Whitt (1992), we obtain

$$\psi(u) = \frac{2e^{au}}{\pi} \int_0^\infty \Re(L_\psi(a+iy)) \cos uy \, dy, \tag{2.3.2}$$

where $i^2 = -1$, $\Re(\cdot)$ is the real part of $(\cdot)$, and a is chosen such that $L_\psi(s)$ is continuous when $s \geq a$. The integral in (2.3.2) can be numerically evaluated using, say, the trapezoidal rule for approximation.

2.4 Dependence in the Risk Model

It is only recently that dependence in the collective risk model is studied in the literature. Gerber, one of the earlier authors in the field, credited Alastair Longley-Cook for providing him with the stimulus for his study, after questioning "that the actuarial world is flat, i.e. that the surplus process has independent increments" (Gerber 1982, 184).

Gerber (1982) started his investigation by considering the probability of ruin when the claim amounts X_i follow a linear (ARMA) model. He proved that a variation of Theorem 2.1.4 holds even when the X_i follows a ARMA(p, q) process, albeit with that restriction that the X_i are bounded, such that $X_i \leq \omega$ for some ω.

Promislow (1991) extended Gerber's work by removing the boundedness condition on X_i, and proved that the variation of Theorem 2.1.4 holds when the X_i are unbounded. He further showed that the theorem would still be correct if we replace τ for some other event involving the surplus process other than ruin.

Nyrhinen (1998) used large deviations theory to derive Lundberg-like bounds on $\psi(u)$ for general models of $S(t)$, as long as $S(t)$ has a moment generating function. Here $S(t)$ can contain a dependence structure. Müller and Pflug (2001) used Markov inequalities to derive similar results. Because of the requirement that $S(t)$ must have a moment generating function, the results in these two papers are only useful for light-tailed aggregate claim distributions. Asmussen et al. (1999) considered bounds for $\psi(u)$ when $N(t)$ has stationary (but not necessarily independent) increments when the claim sizes come from a subexponential distribution.

Cossette and Marceau (2000) studied ruin probabilities for the discrete-time model, where dependence exists in the claim number process representing different classes of business, specified firstly by a Poisson shock model, then later by a negative binomial model that contains a common component. They have shown the probability of ruin increased and the adjustment coefficient decreased under dependence. This is further confirmed by Yuen and Wang (2001), who worked with a continuous-time model, with a dependence structure specified by a thinning model applied to the claim number process. Here a claim from one class of business can generate additional claims for another class of business. They again have found that under this dependence structure, the probability of ruin is greater under dependence than under independence. Similar results were shown through simulation by Albrecher and Kantor (2002) where a Markovian dependence structure is placed on the claim size distribution.

Juri (2002) studied the effect of dependence in the supermodular sense on the adjustment coefficient r. He has shown that the adjustment coefficient decrease as supermodular dependence amongst the X_i increases.

As stated earlier, it is known that the collective risk model with Poisson distributed claim numbers is a good approximation to the individual risk model. Goovaerts and Dhaene (1996) have shown that this is still true for individual risk models with local dependencies, as long as the claim amounts are still independent to the probability of claim. They derived bounds for the total variation distance between the aggregate claims distributions under the dependence model and its compound Poisson approximation. This is followed by Denuit et al. (2002), who derived similar bounds for the total variation distance for the claim number distributions. Some possible forms of dependence within the individual risk model were introduced by Cossette et al. (2002), including a section on the copula model.

CHAPTER 3

DEPENDENT CLAIMS—A SIMULATION STUDY

We now examine what effects dependence has on the ruin scenario. Specifically, we investigate the effects on the probability of ruin and the time to ruin when a dependence structure, specified by a copula, is placed on claim occurrence, whilst keeping the claim sizes independent. Because of the mathematical intractability of solving this problem analytically, we use Monte Carlo simulation, on an individual risk model, to perform the investigation. A large number of trajectories of the surplus process, using different values of initial surplus and copula parameters, and tracked each trajectory until either ruin or the process is censored. Both the number of trajectories that have ruined and the time that ruin occurred are recorded such that the probability of ruin and the distribution of the time to ruin can be derived.

This chapter is organised as follows. Firstly, we review several measures of association and the concept of a copula to provide some preliminary background to our investigation. We outline the simulation procedure in section 3.3. Finally, we discuss the results from the simulation and evaluate the model in section 3.4.

3.1 Preliminary: Measures of Association

Informally, a measure of association is used to quantify the relationship between a pair of random variables (X, Y). That is, we would desire that a measure of association to be positive if it is likely that large (small) values of X would pair up with large (small) values of Y, a situation known as concordance. Similarly, we would desire that this measure to be negative if it is likely that large (small) values of X would pair up with small (large) values of Y, a situation known as discordance. If there is no such relationship between X and Y we would desire that the measure be zero (although the converse do not have to be true). A more formal definition on measures of association, in particular, measure of concordance and measure of dependence, can be found in Nelsen (1999, 136, 170). This section aims to introduce several measures of association that would be used in our study, others can be found in Schweizer and Wolff (1981).

For completeness, we define Pearson's correlation coefficient for two random variables X and Y to be

$$\rho_{XY}^{P} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}, \tag{3.1.1}$$

where $\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$.

Pearson's correlation coefficient is a measure of *linear* association between two random variables, and so is generally not a good measure of association for non-normally distributed random variables. It would be possible to construct X and Y that are dependent but have $\rho_{XY}^{P} = 0$, see for example Embrechts et al. (2002).

3.1.1 Kendall's Tau

Kendall's tau is first discussed in Kendall (1938). For X and Y being two random variables with joint distribution $F_{XY}(x,y)$, Kendall's tau is defined as

$$\tau_{XY} = 4 \iint_{\mathbb{R}^2} F_{XY}(x,y)\, dF_{XY}(x,y) - 1. \tag{3.1.2}$$

An intuitive explanation of Kendall's tau is as follows. Suppose we draw two samples (X_1, Y_1) and (X_2, Y_2) from F_{XY}. Then Kendall's tau is a measure of the probability of concordance between the X and the Y, less the probability of discordance, or

$$\tau_{XY} \propto P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0]. \tag{3.1.3}$$

3.1.2 Spearman's Rho

For X and Y being two random variables with margins $F_X(x)$ and $F_Y(y)$ respectively, and joint distribution $F_{XY}(x,y)$, Spearman's Rho is defined as

$$\rho_{XY} = 12 \iint_{\mathbb{R}^2} F_X(x)F_Y(y)\, dF_{XY}(x,y) - 3. \tag{3.1.4}$$

with the -3 being a normalisation constant.

The type of dependence that is quantified by Spearman's rho is different from that by Kendall's tau. This can be shown using the following intuitive illustration. Instead of drawing both (X_1, Y_1) and (X_2, Y_2) from $F_{XY}(x,y)$, only (X_1, Y_1) is drawn from $F_{XY}(x,y)$. (X_2, Y_2) is drawn from the marginals such that X_2 and

Y_2 are independent. Then Spearman's rho corresponds to the probability of concordance and discordance in (3.1.3).

3.2 Preliminary: Copulas

Copulas are functions that can be used to augment several marginal distributions to form a multivariate distribution, with the dependence structure amongst these marginals specified by the form of the copula. Informally, a copula is a multivariate distribution function with uniform marginals. A more formal definition will be given later.

Whilst a considerable amount of knowledge has already been developed on the subject (see, for example, the introduction by Nelsen [1999]), it has only been relatively recently that the usage of copulas appeared in actuarial literature. Here, papers by Frees and Valdez (1998) and Wang (1998) are a useful introduction to the possible applications of copula theory to actuarial work.

An n-copula is formally defined as a function $C : [0,1]^n \to [0,1]$ such that

1. C is grounded, i.e. $C(\mathbf{x}) = 0$ for all $\mathbf{x} = (x_1, x_2, \ldots, x_i, \ldots, x_n)^T$ with at least one of the $x_i = 0$;
2. C is n-increasing, i.e. $\triangle_{\mathbf{a}}^{\mathbf{b}} C(\mathbf{x}) \geq 0$ for all $\mathbf{a} < \mathbf{b}$; and
3. the one-dimensional margins of C satisfy $C_i(x) = x$ for all $x \in [0,1]$ and $i = 1, 2, \ldots, n$.

When $n = 2$, the usual term used is just "copula".

The connection between a copula and a multivariate distribution is given by Sklar's Theorem.

Theorem 1 (Sklar's Theorem). *Let H be an n-dimensional distribution function with margins $F_1, F_2, \ldots, F_n$. Then there exists an n-copula C such that for all $\mathbf{x}$ in $\mathbb{R}^n$*

$$H(x_1, x_2, \ldots, x_n) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)).$$

Further, if $F_1, F_2, \ldots, F_n$ are all continuous, then C is unique, otherwise C is unique over $\mathrm{Ran}F_1 \times \mathrm{Ran}F_2 \times \cdots \times \mathrm{Ran}F_n$*. The converse of the above is also true.*

Proof. A proof is given by Sklar (1959), which also appeared in Sklar (1996). □

We will now provide two copulas that will be used in our study.

1. The multivariate Fréchet upper bound is specified by

$$M(\mathbf{u}) = \min(u_1, u_2, \ldots, u_n). \tag{3.2.1}$$

   It can be shown (e.g. Nelsen 1999) that for all n-copulas C and for all $\mathbf{u} \in [0,1]^n$,

$$C(\mathbf{u}) \leq M(\mathbf{u}).$$

2. The multivariate Frank's copula is specified by

$$C_\theta(\mathbf{u}) = -\frac{1}{\theta}\left(1 + \frac{\prod_{i=1}^{n}(e^{-\theta u_i} - 1)}{(e^{-\theta} - 1)^{n-1}}\right), \tag{3.2.2}$$

   with $\theta > 0$. (For $n = 2$, θ can also take values less than zero.)

   A different parameterisation of the copula is to let $\eta = e^{-\theta}$ and so

$$C_\eta(\mathbf{u}) = -\frac{1}{\log \eta}\left(1 + \frac{\prod_{i=1}^{n}(\eta^{u_i} - 1)}{(\eta - 1)^{n-1}}\right), \tag{3.2.3}$$

with $0 < \eta < 1$. This parameterisation is particularly useful for simulation purposes.

The Frank copula, first documented in Frank (1979), belongs to a class called Archimedean copulas, which have certain useful properties. Further details can be found in chapter 4 of Nelsen (1999).

Schweizer and Wolff (1981) and Nelsen (1999) give the following copula forms of Kendall's tau and Spearman's rho:

$$\tau_C = 4 \iint_{[0,1]^2} C(u,v)\, dC(u,v) - 1 \tag{3.2.4}$$

$$\rho_C = 12 \iint_{[0,1]^2} C(u,v) - uv\, du\, dv - 3. \tag{3.2.5}$$

3.3 Numerical Simulation

Monte Carlo simulation was used to investigate the effect of dependence has on ruin probability and the time to ruin. The procedure used for the simulation is briefly described below, with a full explanation of the algorithm, and the MATLAB code used to perform the simulation, appearing in the appendix.

For each trajectory in the simulation, we used an individual risk model of 10 000 policies. We start off with an initial surplus u, which is a parameter in our model. Next, claims and premiums are generated. Claims are generated such that the (marginal) probability of occurrence of a claim for each policy is Bernoulli distributed with $q = 0.000\,837$ per time period, and the claim amounts are i.i.d. exponential random variables with a rate of 1. Premiums are charged

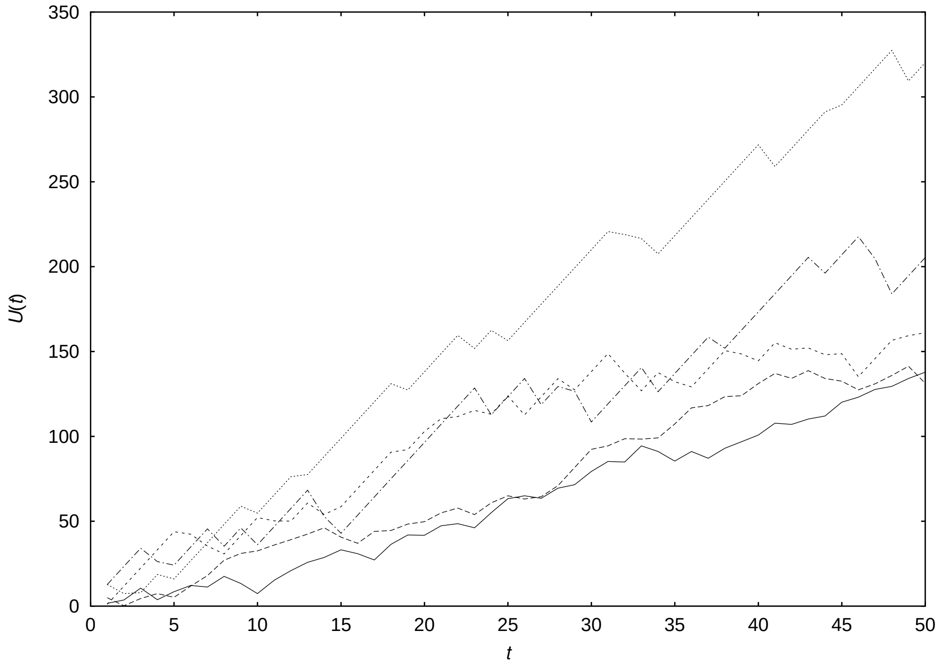

Figure 3.1: Typical trajectories of the surplus process

at a rate of 0.001 075 per policy per time period, representing a safety loading of 28.44%.

This process is then followed until either ruin occurs or $t^+ = 1200$ time periods, whichever is earlier. Those trajectories that reach the right censor t^+ is deemed to have never ruined. Given the distribution of the ruin times as described in the next section, the right censor was applied to catch practically all of the trajectories that ruin at all. This process is then repeated for 1000 replications for each choice of parameter. Figures 3.1 and 3.2 provides some typical trajectories from the simulation, with Figure 3.2 showing trajectories that led to ruin.

The safety loading was selected so that the probability of ruin is relatively

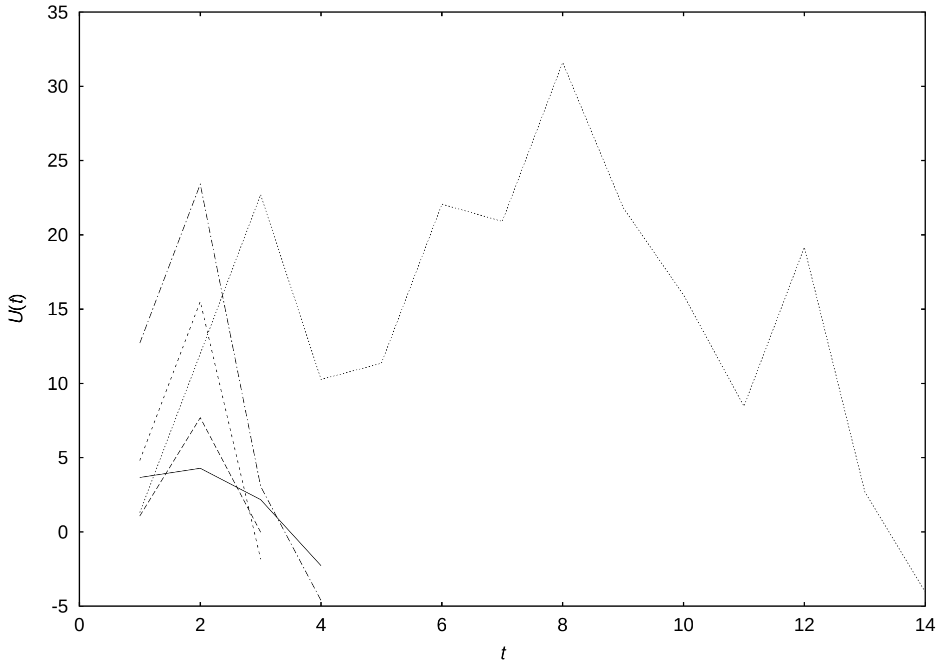

Figure 3.2: Typical trajectories of the surplus process that have ruined

close to one-half. The reason for this choice is related to the variance of the estimate of ψ. Because $\hat{\psi} = l/L$ (where l is the number of replications where ruin occurs and L is the total number of replications) and $l \sim \text{Bin}(L, \psi)$, we have

$$\text{Var}(\hat{\psi}) = \frac{\psi(1-\psi)}{L}.$$

It can then be seen that the relative error of the estimator is minimised when $\psi = 1/2$.

To this model, then, a dependence structure is introduced on the claim occurrence indicator variables, similar to the approach done by Cossette et al. (2002). The Frank copula was used to model the dependence in the claim occurrence,

η	Kendall's τ	Spearman's ρ
0.100	0.243	0.359
0.200	0.174	0.259
0.400	0.101	0.151
0.900	0.012	0.018

Table 3.1: Copula parameters used in the simulation.

with the parameters chosen and their Kendall's tau listed in Table 3.1.

3.4 Results and Discussion

3.4.1 Probability of Ruin

The principal results are summarised in Figures 3.3 and 3.4. Here it can clearly be seen that the probability of ruin increases when a (positive) dependence structure is placed on claim occurrence. In addition, it can be seen that, at least for large amounts of initial capital, the probability of ruin increases as the level of dependence (as measured by Kendall's tau) increases.

Included in Figures 3.3 and 3.4 is the Lundberg upper bound from (2.2.1). The adjustment coefficient is calculated from (2.1.3), using the compound Poisson approximation to the individual model as described in Goovaerts and Dhaene (1996). It can be seen that even for relatively mild levels of positive dependence, the probability of ruin as a function of initial capital exceeds the Lundberg bounds, suggesting that the compound Poisson is *not* a good approximation for an individual model with *global* dependence as modelled here.

Consequently it would be interesting to examine the distribution of N, the total number of claims in a time period. Figures 3.5 and 3.6 show the distribution of

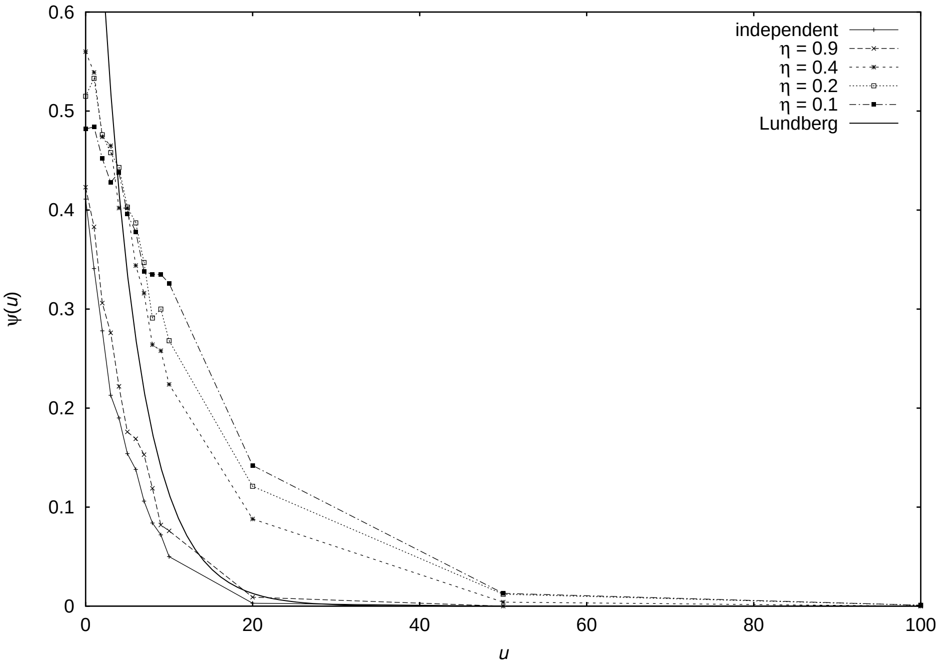

Figure 3.3: Plot of ruin probability against initial capital for independent and Frank-copula models.

the number of claims in a time period for 10 and 50 policies respectively, for different dependence structures on claim occurrence, with the probability of a claim being 0.5. Whilst for lower levels of dependence, the distribution of total number of claims can be sufficiently estimated by the binomial or Poisson distribution, for larger levels of dependence this is clearly not the case.

This can be shown by examining the total variation distance between the distribution of N and a Poisson distribution with the same mean, calculated using the Chen-Stein method as described in Arratia et al. (1989). This is equivalent to examining the goodness of fit of a compound Poisson model to the risk process.

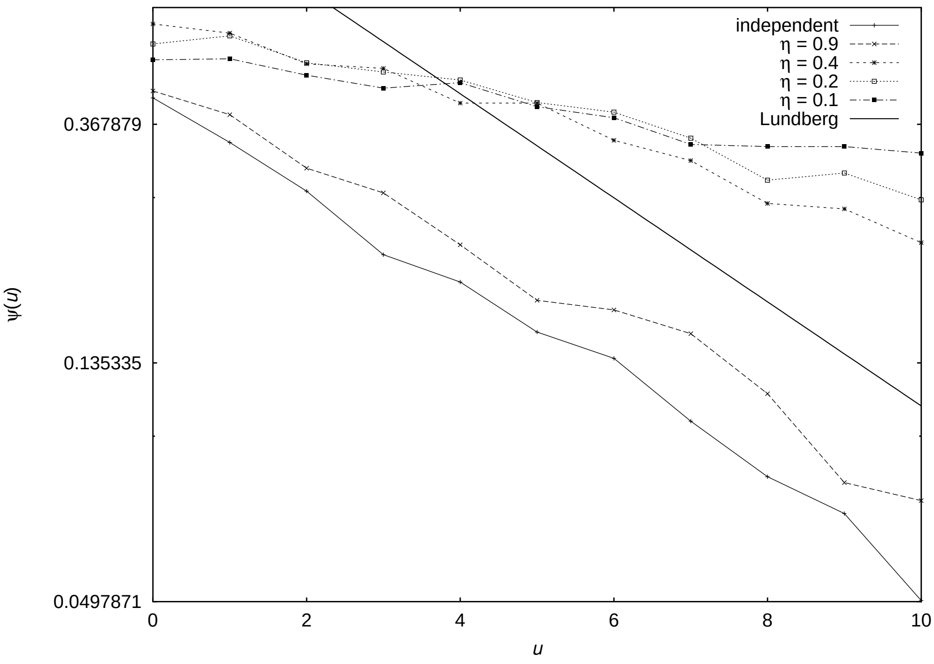

Figure 3.4: Plot of ruin probability against initial capital for independent and Frank-copula models (log scale).

From Goovaerts and Dhaene (1996), we know that the total variation distance D for this model is

$$D \leq (b_1 + b_2)\frac{1 - e^{-\lambda}}{\lambda}, \tag{3.4.1}$$

where

$$\begin{aligned} b_1 &= \sum_{i=1}^{N}\sum_{j=1}^{N} P(X_i = 1)P(X_j = 1) = n^2p^2 = \lambda^2, \\ b_2 &= \sum_{i=1}^{N}\sum_{i\neq j} E(X_iX_j) = n(n-1)[1 - C(p, p, 1, \ldots, 1)]. \end{aligned}$$

Thus $D \to \infty$ when $N \to \infty$, indicating that for large N the compound Poisson is not a good approximation for the individual model.

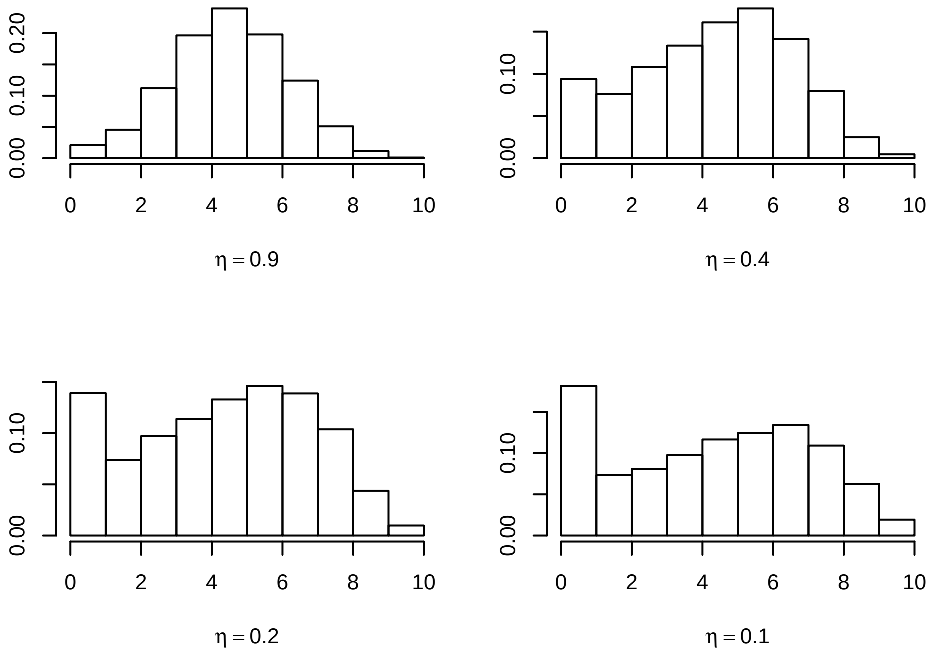

Figure 3.5: Distribution of number of claims for 10 policies

3.4.2 Time to Ruin

We are also interested in the effects a dependence structure has on the time to ruin $\tau < \infty$. Figures 3.7 and 3.8 show the distribution of the time to ruin over different sets of initial capital, for the independent and the $\eta = 0.1$ cases. Results for other dependence structures can be found in the appendix, with a summary appearing in Table 3.2.

The distribution of the time to ruin is concentrated on the first few time periods. This is expected as the surplus from good experience has not been built up over the first few time periods to absorb any adverse experience over that time. Again, the time to ruin tends to increase as the initial capital increases, as the

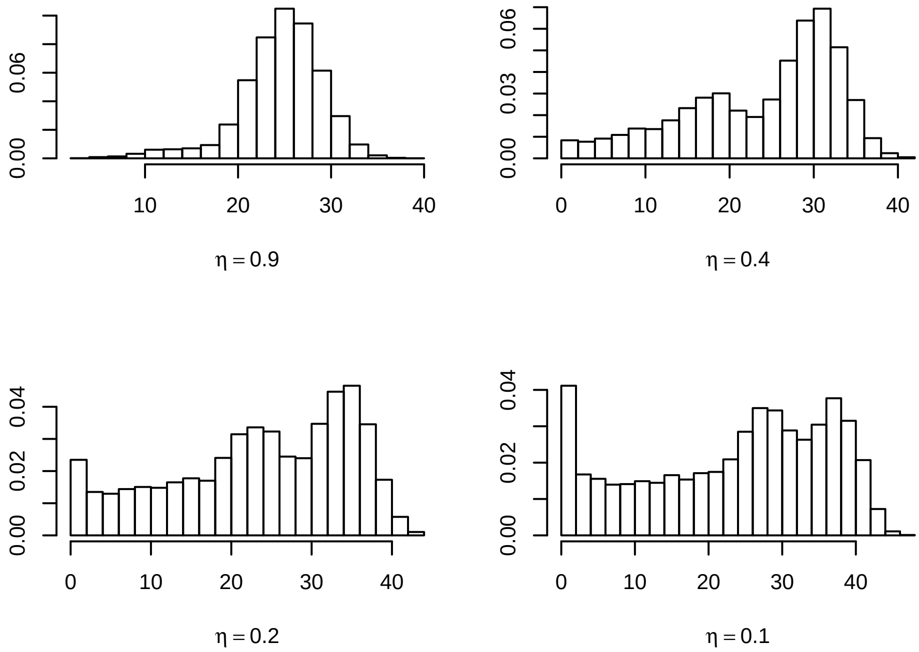

Figure 3.6: Distribution of number of claims for 50 policies

initial capital is sufficient to offset any adverse experience over the first few time periods.

Whilst the distribution of the time of ruin is very similar across all dependence structures, one would notice that the tails of the time to ruin distribution slightly increases as the level of dependence increases. Perhaps this can be attributed to the distribution of N as the level of dependence increases. As the tail of N becomes heavier, so the probability of a high value of N increases. This means that the surplus built up over the first few years may not be sufficient to cover the high number of claims as it would have been under independence. Consequently, ruin is still possible much later during the realisation of the process that in the

	Initial Surplus $u_0 = 0$					Initial Surplus $u_0 = 2$				
	Dependence (η)					Dependence (η)				
Statistics	1.0	0.9	0.4	0.2	0.1	1.0	0.9	0.4	0.2	0.1
Number	207	332	560	541	479	140	233	474	476	473
Mean	2.1	2.1	2.0	2.5	2.5	2.2	2.3	2.5	2.2	2.5
Std Dev	2.2	2.6	2.6	3.8	3.0	2.0	2.3	3.5	3.2	3.8
Minimum	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
Maximum	12.0	29.0	25.0	42.0	27.0	12.0	18.0	34.0	32.0	41.0
	Initial Surplus $u_0 = 5$					Initial Surplus $u_0 = 20$				
	Dependence (η)					Dependence (η)				
Statistics	1.0	0.9	0.4	0.2	0.1	1.0	0.9	0.4	0.2	0.1
Number	78	138	402	404	388	3	7	88	121	142
Mean	3.8	3.5	3.3	2.8	2.6	12.3	9.4	8.3	6.0	4.7
Std Dev	3.5	3.2	3.9	3.6	3.0	6.0	5.9	8.5	5.2	5.2
Minimum	1.0	1.0	1.0	1.0	1.0	6.0	2.0	1.0	1.0	1.0
Maximum	17.0	19.0	38.0	26.0	19.0	18.0	17.0	52.0	32.0	34.0

Table 3.2: Summary statistics of time to ruin

independent case. However, it must be noted that these longer tails may be due to simulation error rather than any systematic changes to the time of ruin.

3.4.3 Assessment of the Model

The beauty of this model lies in the usage of a copula to specify the dependence structure in claim occurrence. All the information regarding the dependence within the claims can be embedded in one copula function. This gives the model great versatility as different dependence structures can be easily incorporated into the model. Thus one can easily examine the effects of different dependence structures on ruin.

Another advantage of this model is its ability to incorporate policy growth. One can easily implement a procedure in algorithm 1 to change the number of policies at the end of each time period. Since we are only using multivariate

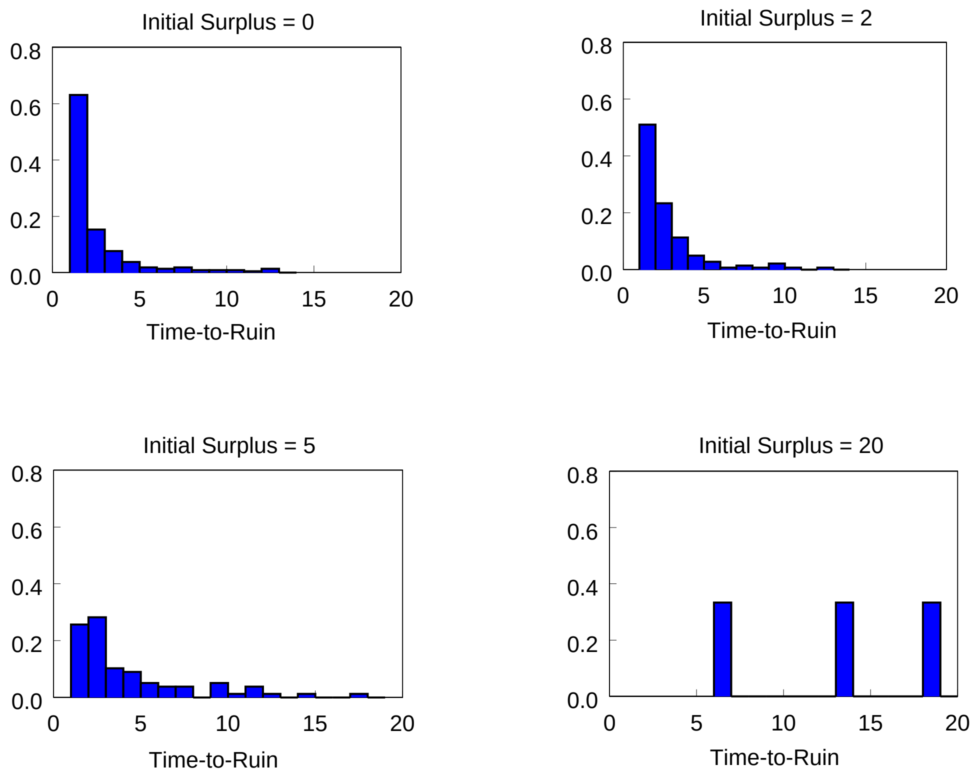

Figure 3.7: Distribution of the time to ruin with independent claims

copulas, the copula model would still be valid even under a change in the number of variables that would be passed through the copula function.

However, this model is not without its shortcomings. Because the dependence structure is applied on claim occurrence, we are in effect analysing the dependence between distribute (Bernoulli) random variables. Marshall (1996) has shown that the copula representation for dependent discrete random variable is not unique. That is, it is possible to describe the same dependence structure within a set of discrete random variable by more than one copula. This means

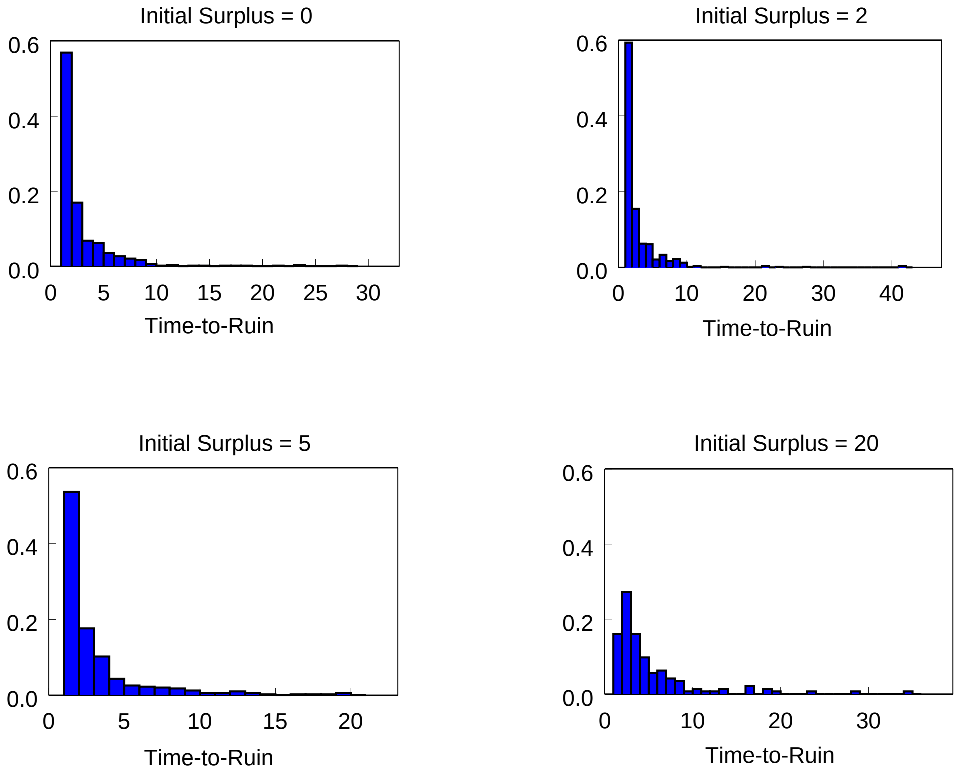

Figure 3.8: Distribution of the time to ruin with $\eta = 0.1$

that estimating the copula form of the dependence structure would be difficult. In addition, in this model the copula is applied to the uniform random variables that are used to generate Bernoulli outcomes, and not on the Bernoulli random variable themselves. Perhaps the copula-like form developed by Tajar et al. (2001), which could be applied directly on the Bernoulli marginals, would be more useful here.

Chapter 4

Conclusion

In this contribution we have examined ruin probabilities when the claim occurrences are not independent. We have used a copula function to specify the dependence structure within the claim occurrences, whilst leaving the claim amounts independent of both the claim occurrence and other claim amounts. This reflect the real-world situation where the occurrence of a single claim may include the occurrence of other claims in the same time period.

Our simulation results confirm intuitive thinking that the probability of ruin increases once a (positive) dependence structure is placed on claim occurrence. The probability of ruin is shown to increase as the level of dependence increases. Moreover, we have shown that under a sufficiently strong dependence structure, the probability of ruin *exceeds* the Lundberg upper bound, if the adjustment coefficient for that bound were calculated under the assumption of independence. Given ruin occurs, the time to ruin does not vary greatly for different levels of dependence.

It must be noted that our results are extremely preliminary. Further studies need to be undertaken to examine the sensitivity of these results for different dependence structures, and specifically for different copulas. Also, given that the results have shown that the Lundberg bound can be exceeded if the adjustment coefficient were calculated assuming independence, it would be useful to quantify the effect of different dependence structures on the adjustment coefficient, and so derive new Lundberg-type bounds for these surplus processes having dependent claims. A suggested approach on how this could be done is given in the section below.

4.1 Comonotonic Bounds for Ruin Probabilities

Firstly we shall introduce the concept of comonotonicity. A random vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)^T$ is comonotonic if the X_i can be expressed as $X_i = F_i^{-1}(U)$, where $U \sim U[0,1]$ and the F_i^{-1} are arbitrary inverse marginal distribution functions. If $\mathbf{X}$ is comonotonic, then the copula resulting from the joint distribution of $\mathbf{X}$ is precisely the Fréchet upper bound described in (3.2.1). An introduction to the theory of comonotonicity can be found in Dhaene et al. (2001).

Consider the set of random variables I_i, having an arbitary dependence structure (including independence), with Bernoulli margins from (1.1.2). In addition, consider I_i^c, which has the same Bernoulli margins as the I_i, but suppose the vector $(I_1^c, I_2^c, \ldots, I_n^c)$ is comonotonic. Theorem 6 of Dhaene et al. (2001) shows that if we define $I_*^c = \sum_i I_i^c$, then $F_{I_*^c}^{-1}(x) = \sum_i F_{I_i^c}^{-1}(x)$. Hence, it then can be seen that with probability p, all the I_i^c are equal to one, and with probability $1-p$, all the I_i^c are equal to zero. Thus if the I_i in (1.1.2) are replaced with the I_i^c, then the entire comonotonic portfolio of policies would act as if it were only one policy: either all the policies in the portfolio claim during one time period, or none of them claims.

Further, if we define $I_* = \sum_i I_i$, then Denuit, Lefèvre, and Utev (2002) have shown that $I_* \leq_{\text{cx}} I_*^c$, where $\leq_{\text{cx}}$ denotes convex (or stop-loss) ordering. Adopting the notation from (1.1.1), then from Frostig (2001) we have $S(t) \leq_{\text{cx}} S^c(t)$. Thus the comonotonic portfolio could be used as an upper bound.

If the claim size of each individual claim is exponentially distributed with parameter α, then it is easily seen that the total claim amount in a time period $S^c(t)$ is either the sum of n exponentially distributed random variables (which is equivalent to one Erlang(α, n) distributed random variable) with probability p or zero with probability $1 - p$. Thus we can now construct an equivalent aggregate claims process

$$S^c(t) = I^c_* X^c,$$

where I^c_* is Bernoulli(p) distributed, and X^c is Erlang(α, n) distributed. The ruin probabilities of this can then be found in closed form through the method described by Dufresne (2001).

Appendix A

Algorithms and Further Results

A.1 Simulation Methods

A.1.1 Individual Model

The following is a generic algorithm used to derive the results presented in this paper. Detailed algorithms on how non-uniform random variates are generated can be found in Devroye (1986).

Algorithm 1. The first few steps involve initialising several variables.

1. Set $n \leftarrow 10\,000$, the number of policies that will be modelled.

2. Let $q \leftarrow 1 - (1 - 0.01)^{1/12}$. Here q represents the probability of a claim per time period, which for convenience we will call "month". The value of q is set such that the probability of claim in 12 months is 0.01.

3. Depending on the model used, the parameters for the claim size distribution F_X are set.

4. Let $\pi \leftarrow nqE(X)(1 + \kappa)$, where π is the premium to be received from the n policies per time period, $E(X)$ is the expected claim amount, and κ is the risk loading.

5. Initialise u, the surplus at time t with the value of the initial reserve.

6. Initialise $t \leftarrow 0$, the time elapsed.

7. Set the right censor at $t^+ \leftarrow 1\,200$.

8. Set $R \leftarrow 0$, the number of trajectories that have incurred ruin.

The main loop will simulate the trajectory of the surplus process by generating claims and receiving premiums until either (a) $U < 0$, indicating ruin, or (b) $t \geq t^+$ where the right censor has been reached.

9. Generate $\mathbf{b} = (b_1, b_2, \ldots, b_n)^T$, an $n \times 1$ vector with uniform marginals. Depending on the model, a dependence structure specified by the Frank copula can be generated within the elements of $\mathbf{b}$. The method of generating outcomes from these copulas can be found in Algorithm 2.

10. Count the number of claims for this period $N \leftarrow \sum_{i=0}^{n} \mathbb{1}_{\{b_i < q\}}$.

11. If $N = 0$, go to 15.

12. Generate the claim amounts $\mathbf{x} = (x_1, x_2, \ldots, b_N)^T$, an $N \times 1$ vector with marginals from F_X.

13. Calculate the aggregate claim for this period $x \leftarrow \sum_{i=1}^{N} x_i$.

14. $u \leftarrow u - x$, with the payment of claims decreasing the surplus.

15. $u \leftarrow u + \pi$, with the receipt of premiums increasing the surplus.

16. $t \leftarrow t + 1$, elapse the time by one time period.

17. Test for ruin. If $u \leq 0$, then ruin has occurred. Set $r \leftarrow r + 1$, and return to 1 to start a new trajectory.

18. Test for censorship. If $t \geq t^{+}$, then the trajectory is right-censored, returning to 1 to start a new trajectory.

19. Otherwise, return to 9 to simulate the next time period.

A.1.2 Generation of Multivariate Copula Outcomes

Generation of bivariate Frank outcomes is relatively simple as this copula belong to the Archimedean family of copulas. Nelsen (1999) used this property to provide several algorithms that would generate bivariate outcomes from Archimedean copulas.

However, generation of multivariate outcomes for these copulas is more difficult. While Frees and Valdez (1998) suggested a recursive algorithm that would generate Frank outcomes, this is inefficient for our purposes.

A more efficient algorithm has been devised firstly by Marshall and Olkin (1988), then by Wang (1998). They recognised that the Frank copula can be constructed from a frailty model. Marshall and Olkin (1988) have shown that the Frank copula can be constructed from a frailty following the discrete logarithmic random variable with parameter $1 - \eta$, and η being the parameter in (3.2.3).

Algorithm 2. 1. Generate r from the frailty distribution. Devroye (1986) provides an algorithm to do this for both the discrete logarithmic distribution with parameter $1 - \eta$ (Frank).

2. Generate $\mathbf{u} = (u_1, u_2, \ldots, u_n)^T$, i.i.d. $U[0, 1]$ variates.

3. $\mathbf{u}^* \leftarrow M_R(r^{-1} \log \mathbf{u})$, where $\log \mathbf{u} = (\log u_1, \log u_2, \ldots, \log u_n)^T$, and M_R is the moment generating function of the frailty. For the discrete logarithmic distribution this is $M_R(t) = \log(1 - (1 - \eta)e^t) / \log \eta$.

4. Return $\mathbf{x} \leftarrow (F_1^{-1}(u_1^*), F_2^{-1}(u_2^*), \ldots, F_n^{-1}(u_n^*))^T$, where the F_i^{-1} are the inverses of the marginal distribution functions.

A.2 MATLAB Code

The following is the MATLAB code used to perform the simulation.

```
%-----------------------------------------------------------
% Main simulation algorithm
function psi = frank(u0, eta)
% Iterations
iteration = 1000;

% Results matrix
% For each iteration, Col 1 = time to ruin
%                     Col 2 = 1 if right censored
res = zeros(iteration, 2);

for i = 1:iteration
   disp(['Iteration: ', int2str(i)]);

   % Number of sub-periods per year (mthly)
   m = 12;
   % Initial number of policies
   policies = 10000;
   % Probability of claim (per year)
   q = 0.01;
   % Probability of claim (mthly)
   qm = 1 - ((1 - q) ^ (1 / m));
   % Claim amount exponential parameter
   lambda = 1;
   % Risk loading
   loading = 0.2844;
   % Premium per policy
```

```
   premium = (policies * q) * (1/lambda) * (1 + loading) / policies;
   % Initial reserve
   u=u0;
   % Time (mthly)
   t = 0;
   % Right-censor at (mthly)
   censor = 1200;

   while u >= 0
      % 1. Generate claims vector
      claimsrnd = frankrnd(eta, policies);
      % 2. Find the number of claims
      nclaims = length(find(claimsrnd < qm));
      if nclaims > 0
         % 3. For each claim, generate claim amount vector
         claimamt = exprnd(lambda, nclaims, 1);
         % 4. Total claim amount
         totalclaims = sum(claimamt);
         % 5. Pay claims
         u = u - totalclaims;
      end
      % 6. Receive premiums
      u = u + ((premium / m) * policies);

      if t == censor
         res(i, 2) = 1;
         break;
      end

      t = t + 1;
  end
   res(i, 1) = t;
end

totalcensored = sum(res(:, 2));
display(['Total iterations censored: ', int2str(totalcensored)]);
psi = totalcensored/iteration;

% Write results
filename = [ 'frank-' num2str(eta) '-exp-' int2str(u0) '.csv'];
csvwrite(filename, res);

%------------------------------------------------------------
% Generates Frank-coupled random variates with uniform marginals.
```

```
function ret = frankrnd(eta, m);
% Marshall and Olkin step 1.
% 1. Generate R from frailty (discrete logarithmic)
% This is the second accelerated generator proposed by Kemp (1981)
% as described in Devroye (1988)
r = 1;
x = rand(1);

if x < (1-eta)
y = rand(1);
   q = 1-exp(log(eta).*y);
   if x <= q.^2
      r = floor(1+(log(x)/log(1-eta.^y)));
   elseif x <= q
      r = 1;
   else
      r = 2;
   end
end

% 2. Generate uniforms
u = rand(m, 1);

% 3. Use MGF of frailty
t = (1/r).*log(u);
u_star = (1/log(eta)).*log(1+exp(t).*(eta-1));

ret = u_star;
```

A.3 Further Results

The figures in this section show the distribution of the time to ruin for cases where the dependence parameter η is 0.9, 0.4 and 0.2.

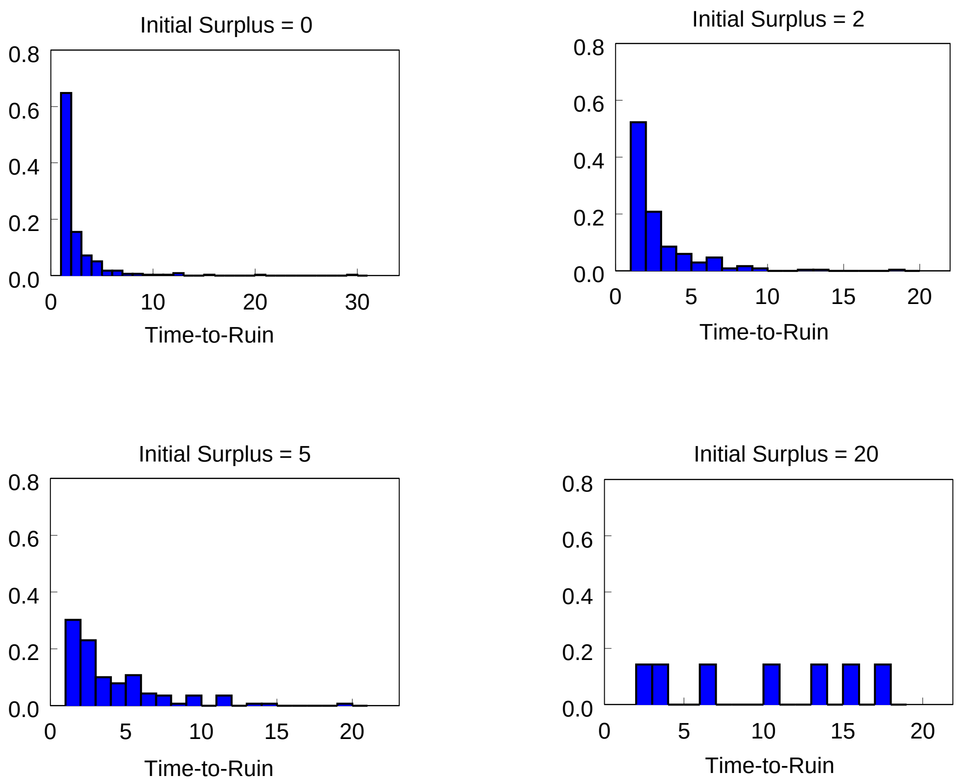

Figure A.1: Distribution of the time to ruin with $\eta = 0.9$

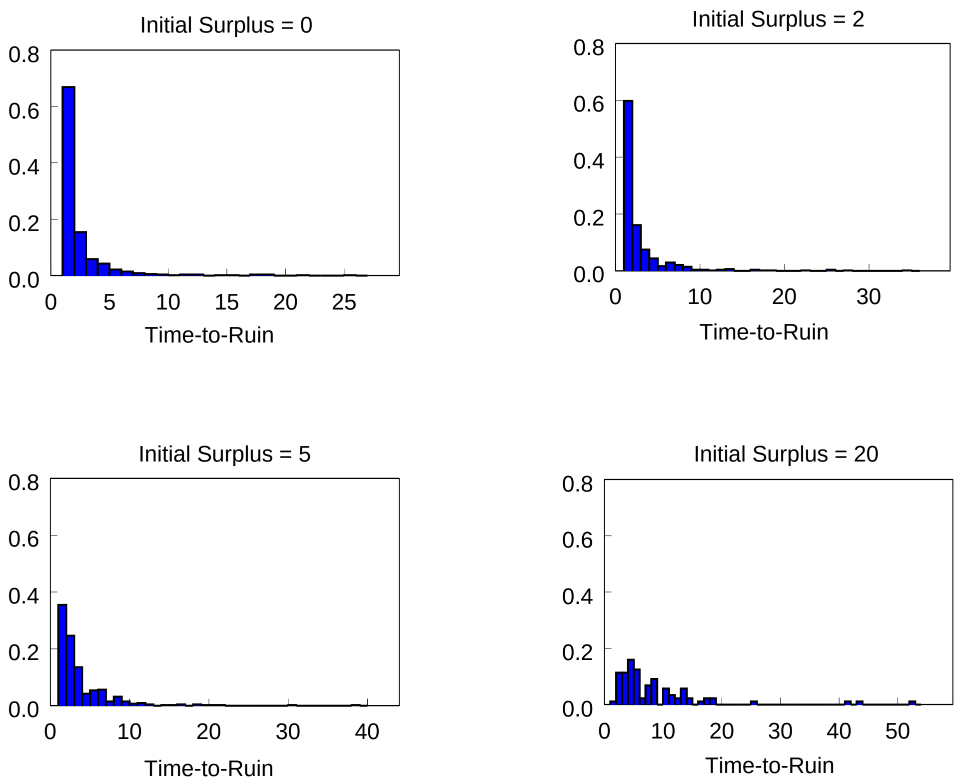

Figure A.2: Distribution of the time to ruin with $\eta = 0.4$

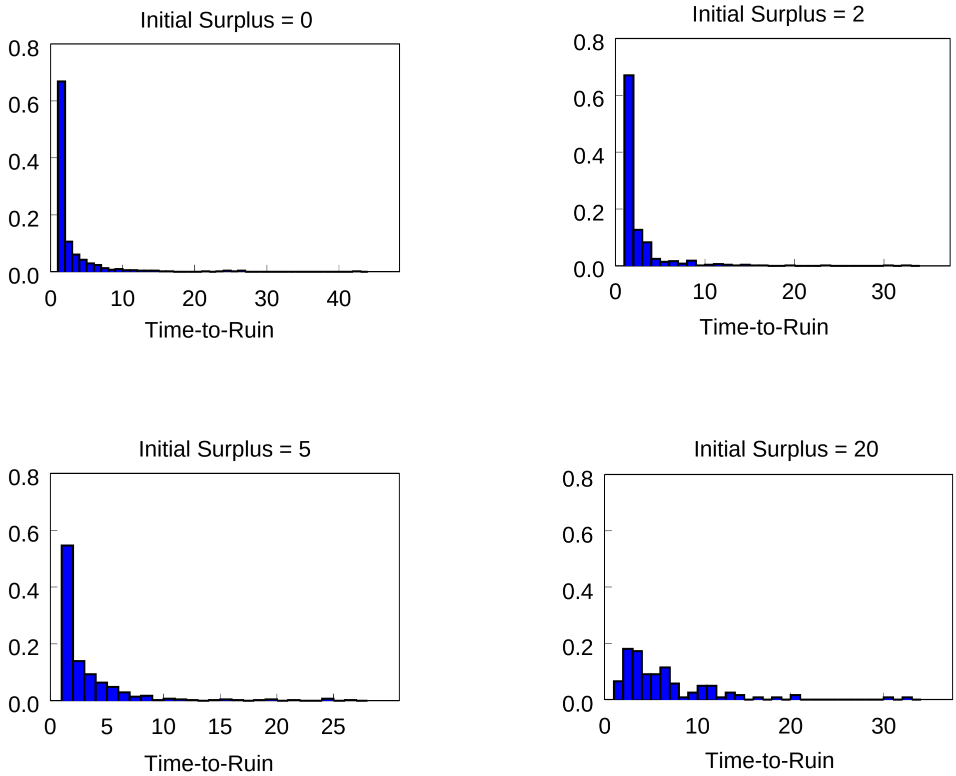

Figure A.3: Distribution of the time to ruin with $\eta = 0.2$

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